

# Probabilistic coding of quantum states

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We discuss properties of probabilistic coding of two qubits to one qutrit and generalize the scheme to higher dimensions. We show that the protocol preserves entanglement between qubits to be encoded and environment and can be also applied to mixed states. We present the protocol which enables encoding of  $n$  qudits to one qudit of dimension smaller than the Hilbert space of the original system and then probabilistically but error-free decode any subset of  $k$  qudits. We give a formula for the probability of successful decoding.

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## I. INTRODUCTION

Qubit in a pure state is described by two real parameters. On the other hand qutrit (quantum system with three-dimensional Hilbert space) in a pure state is described by four real parameters. Thus, the same number specifies two non-entangled qubits in a pure state and one qutrit. Now one can ask the question if it is possible to encode the states of these two qubits to one qutrit. Holevo bound states that using  $d$ -dimensional quantum system (qudit) one can communicate at most  $\log_2 d$  bits of classical information [1]. So it is not possible to achieve the task. However, it has been recently shown both theoretically [2] and experimentally [3] that such encoding is possible in probabilistic, but error-free manner. Namely, the protocol has been proposed [2] which enables decoding with average probability  $2/3$  and perfect fidelity one arbitrarily chosen qubit after the encoding took place. In experimental execution of the protocol the probability of success is  $1/2$ . This is due to fact that in the experiment one replaces POVM with a von Neumann measurement. Bertuskova *et. al* [3] have shown that the protocol can be generalized to higher dimensions. Specifically, they have shown how to encode  $N$  qudits of dimension  $d$  each in one qudit of dimension  $N(d-1)+1$  and then decode one of them. In their scheme both the procedure of encoding and decoding succeeds with a probability strictly less than 1. Here we present the protocol in which the procedure of encoding is always successful (only decoding can be unsuccessful). We also show how one can encode  $n$  non-entangled qudits to one qudit of a dimension smaller than the Hilbert space of the original system and then probabilistically but error-free decode any subset of  $k$  qudits. A simple formula for probability of decoding is given. We also prove that the protocol preserves the entanglement between the qudits to be decoded and the environment. We show that the protocol can be applied to mixed states as well. The paper is organized as follows.

In Section II we review the original scheme and investigate its properties. In Section III we show how one can encode two qudits of dimension  $d$  in one qudit of dimension  $2d-1$  and then probabilistically decode any one of them. In Section IV we present the protocol for encoding many qudits. We also give simple examples illustrating the protocol. The paper ends with a brief summary in Section V.

## II. CODING OF QUBITS ENTANGLED WITH THE ENVIRONMENT AND QUBITS IN MIXED STATES

Let us first briefly describe the original protocol. We introduce two parties Alice and Bob. Alice performs the encoding while Bob tries to decode the qubit with perfect fidelity. We suppose that the states of two qubits are

$$|\Psi_1\rangle = a_1|0\rangle_1 + b_1|1\rangle_1 \quad (1)$$

and

$$|\Psi_2\rangle = a_2|0\rangle_2 + b_2|1\rangle_2. \quad (2)$$

To encode the states of these qubits to one qutrit Alice performs measurement on the joint state of the system  $|\Psi_1\rangle \otimes |\Psi_2\rangle$  given by the following measurement operators

$$M_{0,0} = \frac{1}{\sqrt{3}}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|) \quad (3)$$

$$M_{0,1} = \frac{1}{\sqrt{3}}(|01\rangle\langle 01| + |00\rangle\langle 00| + |11\rangle\langle 11|) \quad (4)$$

$$M_{1,0} = \frac{1}{\sqrt{3}}(|10\rangle\langle 10| + |11\rangle\langle 11| + |00\rangle\langle 00|) \quad (5)$$

$$M_{1,1} = \frac{1}{\sqrt{3}}(|11\rangle\langle 11| + |10\rangle\langle 10| + |01\rangle\langle 01|). \quad (6)$$

It should be emphasized that this is generalized measurement (POVM). If as a result of the measurement Alice

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obtains 0, 0 then the state of two qubits is projected onto a three dimensional subspace and is now given as:

$$|\Psi\rangle = N(a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle), \quad (7)$$

where  $N$  is normalization constant. To recover the first qubit Bob performs a projective measurement given by the following operators:

$$P_{1,S} = |00\rangle\langle 00| + |10\rangle\langle 10| \quad (8)$$

$$P_{1,F} = |01\rangle\langle 01|. \quad (9)$$

If Bob obtains 1,  $S$  as a result of the measurement then the state of the qutrit is projected onto two dimensional subspace and is identical to the state of the first qubit given by Eq. 1. If Bob obtains 1,  $F$  as a result of the measurement then the procedure of decoding fails. Similarly to recover the second qubit Bob performs projective measurement described by the operators:

$$P_{2,S} = |00\rangle\langle 00| + |01\rangle\langle 01| \quad (10)$$

$$P_{2,F} = |10\rangle\langle 10|. \quad (11)$$

It should be stressed that in order to properly choose decoding operators, Bob has to know onto which three-dimensional subspace Alice projected the original state of two qubits. Thus Alice has to send two bits of classical information about the result of the measurement she obtained.

Let us now investigate the possibility of coding the entangled and mixed states. We emphasize that the protocol does not enable encoding of qubits in nonseparable states but this does not reject the possibility of coding qubits entangled with two distinct environments. Thus we suppose that each of Alice's qubits is entangled with a qubit from the environment but they are not correlated (neither quantum nor classically) with each other. We assume that the first qubit and the qubit from the environment are in the pure state:

$$|\Psi_1\rangle = a_1|0\rangle_{E_1}|0\rangle_1 + b_1|1\rangle_{E_1}|0\rangle_1 + c_1|0\rangle_{E_1}|1\rangle_1 + d_1|1\rangle_{E_1}|1\rangle_1. \quad (12)$$

Similarly, the state of the second qubit and the qubit from the environment is:

$$|\Psi_2\rangle = a_2|0\rangle_{E_2}|0\rangle_2 + b_2|1\rangle_{E_2}|0\rangle_2 + c_2|0\rangle_{E_2}|1\rangle_2 + d_2|1\rangle_{E_2}|1\rangle_2. \quad (13)$$

It is convenient to write these states in the following way:

$$|\Psi_1\rangle = |\psi_1\rangle|0\rangle_1 + |\phi_1\rangle|1\rangle_1 \quad (14)$$

$$|\Psi_2\rangle = |\psi_2\rangle|0\rangle_2 + |\phi_2\rangle|1\rangle_2, \quad (15)$$

where

$$|\psi_{1(2)}\rangle = a_{1(2)}|0\rangle_{E_{1(2)}} + b_{1(2)}|1\rangle_{E_{1(2)}} \quad (16)$$

$$|\phi_{1(2)}\rangle = c_{1(2)}|0\rangle_{E_{1(2)}} + d_{1(2)}|1\rangle_{E_{1(2)}} \quad (17)$$

are in general some unnormalized and not necessarily orthogonal vectors. If Alice performs the measurement given by operators of Eqs. 3-6 and obtains for example 0, 0 as the result of the measurement then the state vector of the whole system collapses to

$$|\Psi\rangle = N(|\psi_1\rangle|\psi_2\rangle|00\rangle + |\psi_1\rangle|\phi_2\rangle|01\rangle + |\phi_1\rangle|\psi_2\rangle|10\rangle). \quad (18)$$

To recover the state of the first or the second qubit and the corresponding qubit from the environment, Bob performs projective measurement given by operators of Eqs. 8 and 9 or 10 and 11 respectively. We see that the original protocol preserves the entanglement between the qubit to be recovered and the environment.

Let us now comment on the coding of mixed states. Let us suppose that we have two qubits. The first (second) qubit is in a state described by density matrix  $\rho_{1(2)}$  and the state of the whole system is:

$$\rho = \rho_1 \otimes \rho_2. \quad (19)$$

It is well known that any mixed state can be purified [4]. Thus, we can assume that the mixed state  $\rho_{1(2)}$  is obtained from the pure state of the system and the environment by tracing out the latter. Because the scheme preserves entanglement between the system and the environment, the density matrix of the qubit which is successfully decoded does not change and we conclude that the protocol can be applied to mixed states of the form 19.

### III. CODING OF TWO QUDITS

Let us now describe a generalization of the scheme for coding of two qudits. We assume that we have two non-entangled qudits of the dimension  $d$ . Each of them is in a pure state (The protocol applies to mixed states as well.):

$$|\Psi\rangle = \sum_{i=0}^{d-1} a_i|i\rangle \otimes \sum_{i=0}^{d-1} b_i|i\rangle. \quad (20)$$

To encode the states of these two qudits in one qudit of dimension  $2d - 1$  Alice performs the measurement described by the following operators:

$$M_{i,j} = \frac{1}{\sqrt{2d-1}}(|ij\rangle\langle ij| + \sum_{\substack{k=0 \\ k \neq i}}^{d-1} |kj\rangle\langle kj| + \sum_{\substack{l=0 \\ l \neq j}}^{d-1} |il\rangle\langle il|). \quad (21)$$

These are hermitian operators. Each term  $|ij\rangle\langle ij|$  is present in  $1 + 2(d-1)$  operators, namely in  $M_{i,j}$ ,  $M_{k,j}$  ( $k \neq i$ ) and  $M_{i,l}$  ( $l \neq j$ ) and thus, these operators satisfy the condition.

$$\sum_{i,j=0}^{d-1} M_{i,j}^\dagger M_{i,j} = I. \quad (22)$$

Taking it all together we see that the operators  $M_{i,j}$  are indeed the measurement operators. Each of these operators projects the initial state of two qudits onto  $(2d-1)$ -dimensional subspace of the original Hilbert space. We can now treat our system as a qudit of the dimension  $(2d-1)$ . To decode the state of the first qudit, Bob performs a projective measurement described by the operators:

$$P_{1,S} = \sum_{k=0}^{d-1} |kj\rangle\langle kj| \quad (23)$$

and

$$P_{1,F} = \sum_{l=0, l \neq j}^{d-1} |il\rangle\langle il|. \quad (24)$$

If he obtains 1,  $S$  as a result of the measurement then decoding succeeds, otherwise it fails. The procedure for decoding of the second qudit is similar.

#### IV. CODING OF MANY QUDITS

We now describe the protocol for encoding of  $n$  qudits in such a way that one can probabilistically, but error-free decode any subset of  $k$  qudits. Let us suppose that we have  $n$  non-entangled qudits of dimension  $d$ . Each of these qudits is in a pure state and the state of the whole system is:

$$|\Psi\rangle = \sum_{i=0}^{d-1} a_i |i\rangle \otimes \sum_{i=0}^{d-1} b_i |i\rangle \otimes \sum_{i=0}^{d-1} c_i |i\rangle \dots \quad (25)$$

In order to encode these  $n$  qudits in such a way that Bob can later decode any subset of  $k$  qudits, Alice performs a measurement described by operators

$$\begin{aligned} M_{i,j,k,\dots} = & \frac{1}{\sqrt{D_k}} (|ijk\dots\rangle\langle ijk\dots| + \\ & + \sum_{\substack{p=0, \\ p \neq i}}^{d-1} |pjk\dots\rangle\langle pjk\dots| + \sum_{\substack{q=0, \\ q \neq j}}^{d-1} |iqk\dots\rangle\langle iqk\dots| + \\ & + \sum_{\substack{r=0, \\ r \neq k}}^{d-1} |ijr\dots\rangle\langle ijr\dots| + \dots + \\ & + \sum_{\substack{p=0, q=0, \\ p \neq i, q \neq j}}^{d-1} |pqk\dots\rangle\langle pqk\dots| + \sum_{\substack{p=0, r=0, \\ p \neq i, r \neq k}}^{d-1} |pjr\dots\rangle\langle pjr\dots| + \\ & + \sum_{\substack{q=0, r=0, \\ q \neq j, r \neq k}}^{d-1} |iqr\dots\rangle\langle iqr\dots| + \dots + \\ & + \text{OTHER TERMS}), \end{aligned} \quad (26)$$

where "OTHER TERMS" stands for similar sums over three, four .... and  $k$  indices. The constant  $D_k$  in the

above equation is equal to the dimension of the subspace onto which  $M_{i,j,k,\dots}$  projects and

$$D_k = \sum_{i=0}^k \binom{n}{i} (d-1)^i. \quad (27)$$

Similar arguments as before can be used to show that these operators indeed describe a measurement. Because if  $k < n$  then

$$\begin{aligned} D_k &= \sum_{i=0}^k \binom{n}{i} (d-1)^i < \\ &< \sum_{i=0}^n \binom{n}{i} (d-1)^i = (1 + (d-1))^n = d^n \end{aligned} \quad (28)$$

the qudits are encoded in a system with the Hilbert space of smaller dimension than the original one.

To decode  $k$  qudits Bob performs a projective measurement described by the operators

$$P_S = \sum_{p=0, q=0, \dots}^{d-1} |pqk\dots\rangle\langle pqk\dots| \quad (29)$$

$$P_F = I - P_S, \quad (30)$$

where the sum is taken over indices belonging to the qudits to be decoded and other indices are equal to those which specify the result of the measurement 26. We will also notice that if the qudits to be decoded are entangled between themselves then the procedure succeeds and preserves the entanglement. However the qudits which are decoded cannot be correlated with those which are not decoded.

Let us now illustrate the whole protocol with a simple example. We assume that we have three qubits. Now we can encode them in two different ways: (1) Alice encodes three qubits in such a way that any one of them can be later decoded and (2) Alice encodes three qubits in such a way that any two of them can be later decoded. In both cases the initial state of the system is:

$$\begin{aligned} |\Psi\rangle = & a_1 a_2 a_3 |000\rangle + a_1 a_2 b_3 |001\rangle + \\ & + a_1 b_2 a_3 |010\rangle + a_1 b_2 b_3 |011\rangle + b_1 a_2 a_3 |100\rangle + \\ & + b_1 a_2 b_3 |101\rangle + b_1 b_2 a_3 |110\rangle + b_1 b_2 b_3 |111\rangle. \end{aligned} \quad (31)$$

In the case of the first coding we have  $n = 3$ ,  $d = 2$  and  $k = 1$ . To encode three qubits Alice projects the state of the system on 4-dimensional subspace with measurement operators defined in Eq. 26, for example

$$\begin{aligned} M_{0,0,0} = & \frac{1}{2} (|000\rangle\langle 000| + |001\rangle\langle 001| + \\ & + |010\rangle\langle 010| + |100\rangle\langle 100|). \end{aligned} \quad (32)$$

If Alice obtains 0, 0, 0 as a result of her measurement then the state of the system becomes

$$\begin{aligned} |\Psi\rangle = & N (a_1 a_2 a_3 |000\rangle + a_1 a_2 b_3 |001\rangle + \\ & + a_1 b_2 a_3 |010\rangle + b_1 a_2 a_3 |100\rangle). \end{aligned} \quad (33)$$

If Bob wants to decode the state of the first qubit he performs a projective measurement described by the operators

$$P_{1,S} = |000\rangle\langle 000| + |100\rangle\langle 100| \quad (34)$$

$$P_{1,F} = |010\rangle\langle 010| + |001\rangle\langle 001|. \quad (35)$$

If he obtains 1,  $S$  as a result of the measurement then he successfully decodes the first qubit.

In the case of the second coding we have  $n = 3$ ,  $d = 2$  and  $k = 2$ . To encode three qubits Alice projects the state of the system onto a 7-dimensional subspace with measurement operators defined in Eq. 26, for example

$$\begin{aligned} M_{0,0,0} = & \frac{1}{\sqrt{7}}(|000\rangle\langle 000| + \\ & + |001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100| + \\ & + |011\rangle\langle 011| + |101\rangle\langle 101| + |110\rangle\langle 110|). \end{aligned} \quad (36)$$

It should be noted here that the dimension of the space onto which the original space is projected depends on two things: (1) the number of qudits to be encoded and (2) the number of qudits to be decoded.

If Alice obtains 0,0,0 as a result of her measurement then the state of the system becomes

$$\begin{aligned} |\Psi\rangle = & N(a_1a_2a_3|000\rangle + a_1a_2b_3|001\rangle + \\ & a_1b_2a_3|010\rangle + b_1a_2a_3|100\rangle + a_1b_2b_3|011\rangle + \\ & b_1a_2b_3|101\rangle + b_1b_2a_3|110\rangle). \end{aligned} \quad (37)$$

If Bob wants to decode the state of the first and the second qubit he performs a projective measurement described by the operators

$$\begin{aligned} P_S = & |000\rangle\langle 000| + |100\rangle\langle 100| + |010\rangle \\ & \langle 010| + |110\rangle\langle 110| \end{aligned} \quad (38)$$

$$P_F = |001\rangle\langle 001| + |011\rangle\langle 011| + |101\rangle\langle 101|. \quad (39)$$

If he obtains  $S$  as a result of the measurement then he has successfully decoded the first and the second qubit.

In the protocols described the procedure of decoding is always successful, however one does not know onto which subspace the initial state of the system will be projected. The choice of the decoding measurement depends on (1) the qudits to be decoded and (2) the subspace onto which the initial state was projected. Because of the latter Alice must send to Bob  $k \log_2 d$  bits of classical information about the result of her measurement. The procedure of decoding is probabilistic and it only succeeds with some probability. If we assume that each qudit is prepared in a randomly chosen pure state then the average probability of successful decoding of  $k$  qudits of  $n$  encoded qudits is equal to:

$$p_S = \frac{d^k}{D_k}. \quad (40)$$

This is the dimension of the Hilbert space of  $k$  decoded qudits divided by the dimension of the Hilbert space of the qudit to which  $n$  qudits were encoded.

## V. SUMMARY

We have shown that scheme of Ref. [2] preserves the entanglement between the qubit to be decoded and the environment and can be also used for coding of mixed states. We also presented a much more general protocol which enables encoding of  $n$  qudits in a one qudit of dimension smaller than the dimension of the Hilbert space of the original system and then probabilistically decode any subset of  $k$  of them. The probability of successful decoding is equal to the dimension of the Hilbert space of  $k$  qudits divided by the dimension of the qudit in which  $n$  qudits are encoded.

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